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Quasicomplete factorization and the two machine flow shop problem

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Abstract

A connection is made between the Two Machine Flow Shop Problem (2MFSP) from job scheduling theory and the issue of quasicomplete factorization of rational matrix functions. A quasicomplete factorization is a factorization into elementary (i.e., degree one) factors such that the number of factors is minimal. For a companion based matrix function W, the number of factors in a quasicomplete factorization of W is related in a simple way to the minimum makespan of an instance J of 2MFSP which can be associated to W. As a consequence of this result, variants of the 2MFSP and other types of factorization can be related too. © 1998 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction

Let W be a rational $n \times n$ matrix function satisfying the assumption (standard in this paper) that $W(\infty) = I_n$, where I_n is the $n \times n$ identity matrix. By

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results of DeWilde and Vandewalle [10], such a function always admits a factorization into elementary factors, i.e., a factorization of the form

$$W(\lambda) = \left(I_n + \frac{1}{\lambda - \alpha_1} R_1\right) \cdots \left(I_n + \frac{1}{\lambda - \alpha_N} R_N\right),$$

where $\alpha_1, \ldots, \alpha_N$ are complex numbers, and $R_1, \ldots R_N$ are matrices of rank one. Such a factorization is called *quasicomplete* if the number of elementary factors is minimal for the rational matrix function under consideration. The number of elementary factors arising in a quasicomplete factorization of W is denoted by $\rho(W)$. Using realization theory, Zuidwijk [21,22] described quasicomplete factorizations and reproved the result by DeWilde and VandeWalle. One of the further results in this direction is given by Corollary 5.7 in the present paper.

An important integer associated with a rational matrix function W is the McMillan degree $\delta(W)$ of W: the number of poles of W counted according to pole multiplicity. Elementary rational matrix are rational matrix functions with McMillan degree one. The McMillan degree satisfies a sublogarithmic property, i.e., $\delta(W_1 W_2) \leqslant \delta(W_1) + \delta(W_2)$. Therefore, if $W = W_1 \cdots W_\rho$ is a quasicomplete factorization for W, hence $\rho = \rho(W)$ and $\delta(W_j) = 1$ for $j = 1, \ldots, \rho$, then $\delta(W) \leqslant \sum_{j=1}^{\rho} \delta(W_j) = \rho(W)$.

A quasicomplete factorization $W = W_1 \cdots W_\rho$ is called a *complete* factorization whenever $\rho(W) = \delta(W)$. In fact, a complete factorization is a minimal factorization into elementary factors (cf. [2]). Contrary to the case of quasicomplete factorization, a complete factorization need not exist for a given rational matrix function. These matters will be discussed in more detail in Section 2. We also refer to [2,4,22].

In this paper, we study the connection between quasicomplete factorization and the Two Machine Flow Shop Problem (2MFSP). The latter is a combinatorial job scheduling problem: two machines have to process a number of jobs taking into account certain precedence relations. The aim is to do so in a minimal amount of time, the so-called makespan. In Section 3, we make this more precise.

The connection between quasicomplete factorization and 2MFSP is made via companion based matrix functions. A companion based $n \times n$ matrix function W is a rational $n \times n$ matrix function that admits a minimal realization

$$W(\lambda) = I_n + C(\lambda I_m - A)^{-1} B,$$

where the $m \times m$ matrices A and $A^\times = A - BC$ are first companion matrices. The class of companion based matrix functions is studied by Bart and Kroon [6], who also indicate a connection between the problem of complete factorization of companion based matrix functions and 2MFSP. In particular, they prove that with each companion based matrix function W one can associate an instance J of 2MFSP and vice versa. Furthermore, *if W is a companion* *based matrix function and J is the associated instance of 2MFSP, then W admits complete factorization if and only if* $\mu(J) - 1 \leqslant \delta(W)$. Here $\delta(W)$ denotes the McMillan degree of W and $\mu(J)$ denotes the minimum makespan of J. This result can be viewed as a reformulation of a result on companion matrices that was obtained earlier by Bart and Hoogland [4]; see also [9].

In the present paper, this result is generalized by establishing the following connection between quasicomplete factorization of companion based matrix functions and 2MFSP: *If W is a companion based matrix function and J is the associated instance of 2MFSP, then the number of elementary factors* $\rho(W)$ *occurring in a quasicomplete factorization of W equals* $\max\{\mu(J) - 1, \delta(W)\}$. The latter result indeed generalizes the former, since it implies that a companion based matrix function W admits complete factorization (i.e., $\rho(W) = \delta(W)$) if and only if $\mu(J) - 1 \leqslant \delta(W)$.

The results of the present paper are also closely connected to the results of Bart and Kroon [7]. In the latter paper, a connection is described between minimal factorizations of companion based matrix functions that are optimal in a certain sense and variants of 2MFSP. In particular, it is shown how variants of 2MFSP can be used to find a minimal factorization of a companion based matrix function where the maximum McMillan degree of the factors is minimal, or a factorization with a maximum number of factors. Both variants of 2MFSP receive attention in Section 6 of the present paper in combination with the main result of this paper.

The present paper is organized as follows: In Section 2 we provide material on factorization of rational matrix functions, and in Section 3 we give a description of 2MFSP. In Section 4 we present the main result of this paper already indicated above. In Section 5 we provide its proof. The paper is concluded with Section 6, in which a connection is made between optimization problems from [6,7] and quasicomplete factorization.

It should be noted that most examples in this paper involve companion based 2×2 matrix functions of the type discussed by Bart and Kroon [5]. However, the results of this paper are also valid for companion based matrix functions of arbitrary dimension.

We finish this introduction with some remarks on the notation that is used in this paper. Whenever useful, we shall identify a matrix with its action as a linear mapping relative to the standard bases. For an $n \times m$ matrix B, we denote its nullspace by Ker B and its range by Ran B. The linear span of a set $\mathscr{S}$ of vectors in a given linear space is denoted by span $\mathscr{S}$. If $T = (T_{ij})_{i,j=1}^{m}$ is an $m \times m$ matrix, then $\sigma(T)$ denotes the *spectrum* of T, i.e., the set of eigenvalues of T in the complex plane $\mathbb{C}$. The complement $\rho(T) = \mathbb{C} \setminus \sigma(T)$ of the spectrum is called the *resolvent set*. The $n \times n$ identity matrix is denoted by I_n, while the symbol O stands for a rectangular matrix with zero entries, the size of which will always be clear from the context.

2. Rational matrix functions and factorization

In this section we present some background material on rational $n \times n$ matrix functions that is used in this paper. We also describe complete and quasi-complete factorization of rational matrix functions. It should be noted that throughout this paper all rational $n \times n$ matrix functions are assumed to be analytic at ∞ with value I_n, the $n \times n$ identity matrix. Relevant references are [2,3,10,13,15–17,20].

Let W be a rational $n \times n$ matrix function (which, according to the standing assumption above, is analytic at ∞ with $W(\infty) = I_n$). By a *realization* of W we mean a representation of the form

$$W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B, \tag{1}$$

where A is an $m \times m$ matrix, B is an $m \times n$ matrix and C is an $n \times m$ matrix. It is always possible to find such a representation (cf. [2], and the references given there).

If (1) is a realization of W, then

$$W^{-1}(\lambda) = I_n - C(\lambda I_m - A + BC)^{-1}B \tag{2}$$

is a realization of the rational matrix function W^{-1} given by $W^{-1}(\lambda) = W(\lambda)^{-1}$. It is customary to write $A^{\times}$ for the matrix $A - BC$. With this notation (2) becomes $W^{-1}(\lambda) = I_n - C(\lambda I_m - A^{\times})^{-1}B$.

The smallest possible m for which a given rational matrix function W admits a realization (1) is called the *McMillan degree* of W and is denoted by $\delta(W)$. It equals the total number of poles of W counted according to pole multiplicity. A precise description of the notion of pole multiplicity, which can be found in [2] or [5], is not necessary for a proper understanding of this paper. Note that $\delta(W) = 0$ if and only if $W(\lambda) = I_n$ for all λ.

The realization (1) is called *minimal* if $m = \delta(W)$. Minimal realizations are essentially unique: if (1) is a minimal realization of W, then all minimal realizations of W can be obtained by replacing A, B and C by SAS^{-1}, SB and CS^{-1} respectively, where S runs through the invertible $m \times m$ matrices. This result is known as the *State Space Isomorphism Theorem.*

To facilitate later discussions, we associate two polynomials with a rational matrix function W. The *pole polynomial* p_W and the *zero polynomial* $p_W^{\times}$ of W are defined by

$$p_W(\lambda) = (\lambda - \alpha_1)\cdots(\lambda - \alpha_m), \qquad p_W^{\times}(\lambda) = (\lambda - \alpha_1^{\times})\cdots(\lambda - \alpha_m^{\times}),$$

where $\alpha_1, \ldots, \alpha_m$ are the poles of W counted according to pole multiplicity and $\alpha_1^{\times}, \ldots, \alpha_m^{\times}$ are the zeros of W counted according to zero multiplicity, i.e., $\alpha_1^{\times}, \ldots, \alpha_m^{\times}$ are the poles of W^{-1} counted according to their pole multiplicities. Obviously, both p_W and $p_W^{\times}$ are monic and have degree $m = \delta(W)$. For what follows, it is important to note that, if (1) is a minimal realization of W, then p_W and $p_W^\times$ are the characteristic polynomials of A and $A^\times$ respectively.

The McMillan degree $\delta(W)$ is sublogarithmic in the following sense: If $W = W_1 \cdots W_r$ is a factorization of W, then

$$\delta(W) \leqslant \delta(W_1) + \cdots + \delta(W_r). \tag{3}$$

Of special interest are factorizations with equality in (3). These are factorizations in which pole-zero cancellation does not occur (cf. [2] or [11]). They are called *minimal factorizations*. It should be noted that there exist rational matrix functions of McMillan degree greater than one without any non-trivial minimal factorization.

A rational matrix function is called *elementary* if its McMillan degree equals one. A *complete factorization* is a minimal factorization involving elementary factors only. Thus a factorization of a rational matrix function W is complete if it has the form

$$W(\lambda) = \left(I_n + \frac{1}{\lambda - \alpha_1} R_1\right) \cdots \left(I_n + \frac{1}{\lambda - \alpha_m} R_m\right),$$

where m is the McMillan degree of W, where $\alpha_1, \ldots, \alpha_m$ are the poles of W counted according to pole multiplicity, and where $R_1, \ldots, R_m$ are $n \times n$ matrices of rank 1. Not all rational matrix functions admit complete factorization. Indeed, the rational matrix function

$$W(\lambda) = \begin{pmatrix} 1 & 1/\lambda^2 \\ 0 & 1 \end{pmatrix}$$

has McMillan degree $\delta(W) = 2$, while it is not the product of two elementary factors; for details, see Corollary 5.7. On the other hand, it has been shown in [10] that each rational matrix function is the product of a certain number of elementary factors. This number may exceed the McMillan degree of the function. A *quasicomplete factorization* of a given rational matrix function is a factorization of the function into a minimal number of elementary factors. We denote this minimal number of elementary factors for a given rational matrix function W by $\rho(W)$. In case W admits complete factorization, we get $\rho(W) = \delta(W)$. In all other cases, $\rho(W) > \delta(W)$ holds. In fact, it is known [10] that $\delta(W) \leqslant \rho(W) \leqslant 2\delta(W) - 1$ for all non-trivial rational matrix functions W.

An important tool to factorize a given rational matrix into elementary factors is the notion of complementary triangular forms. A pair of $m \times m$ matrices A, Z *admits simultaneous reduction to complementary triangular forms*, if there exists an invertible $m \times m$ matrix S, such that $S^{-1}AS$ is upper triangular and $S^{-1}ZS$ is lower triangular. We shall not discuss this notion in great detail, but refer to [4,21] instead. An important result of Bart et al. [2] shows how complete factorizations can be described using the notion of complementary triangular forms: A rational matrix function W with minimal realization $W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B$ admits complete factorization if and only if the pair of matrices $A, A^\times$ admits simultaneous reduction to complementary triangular forms. Furthermore, a pair A, Z admits simultaneous reduction to complementary triangular forms if and only if there exist matching maximal invariant nests of subspaces for A and Z respectively. This means that there exist collections $\{M_k\}_{k=0}^m$ and $\{N_k\}_{k=0}^m$ of subspaces in $\mathbb{C}^m$ with the following properties. First, the collections are *maximal invariant nests* for the respective matrices:

$$\dim M_k = \dim N_k = k, \quad M_k \subseteq M_{k+1}, \quad N_k \subseteq N_{k+1},$$

$$AM_k \subseteq M_k \qquad ZN_k \subseteq N_k,$$

for $0 \leqslant k \leqslant m$, where we have assumed that $M_{m+1} = N_{m+1} = \mathbb{C}^m$. In addition, the nests should satisfy the *matching condition*:

$$M_k \oplus N_{m-k} = \mathbb{C}^m, \quad 0 \leqslant k \leqslant m.$$

The following result, which we will use in this paper, can be found in [21,22].

Proposition 2.1. *Let W be a rational matrix function with minimal node $(A, B, C; m, n)$. Then W admits a factorization into ρ elementary factors if and only if $\rho \geqslant m = \delta(W)$ and there exists an integer κ with $0 \leqslant \kappa \leqslant \rho - m$ and a $(\kappa, m, \rho - m - \kappa)$-dilation $(\hat{A}, \hat{B}, \hat{C}; \rho, n)$ of the minimal node $(A, B, C; m, n)$ such that the pair of $\rho \times \rho$ matrices $\hat{A}$ and $\hat{A}^\times = \hat{A} - \hat{B}\hat{C}$ admits simultaneous reduction to complementary triangular forms.*

Here a *node* $(A, B, C; m, n)$ is a quintet containing integers m and n, as well as an $m \times m$ matrix A, an $n \times m$ matrix B, and an $m \times n$ matrix C. If a rational matrix function W admits a realization $W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B$, then $(A, B, C; n, m)$ is a node for W.

A node is *minimal* if the pair of matrices (A, B) is *controllable*, i.e.,

$$\operatorname{Ran} B + \operatorname{Ran}(AB) + \cdots + \operatorname{Ran}(A^{m-1}B) = \mathbb{C}^m,$$

and the pair of matrices (A, C) is *observable*, i.e.,

$$\operatorname{Ker} C \cap \operatorname{Ker}(CA) \cap \cdots \cap \operatorname{Ker}(CA^{m-1}) = (0).$$

A node is minimal for a rational matrix function W if $W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B$ is a minimal realization of the rational matrix function W. In fact, a node is minimal for a rational matrix function W if and only if it is a node for W and if it is a minimal node.

A node $(\hat{A}, \hat{B}, \hat{C}; \rho, n)$ is a *dilation* of the node $(A, B, C; m, n)$ if $m \leqslant \rho$ and there exists an integer κ with $0 \leqslant \kappa \leqslant \rho - m$ such that the matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$ can be written as

$$\hat{A} = \begin{pmatrix} A_1 & A_{12} & A_{13} \\ 0 & A & A_{23} \\ 0 & 0 & A_3 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} B_1 \\ B \\ 0 \end{pmatrix}, \qquad \hat{C} = \begin{pmatrix} C_1 & C & 0 \end{pmatrix},$$

with respect to the decomposition $\mathbb{C}^p = \mathbb{C}^\kappa \oplus \mathbb{C}^m \oplus \mathbb{C}^{p-m-\kappa}$. If the node $(\hat{A}, \hat{B}, \hat{C}; p, n)$ is a dilation of the node $(A, B, C; m, n)$, then $\sigma(A) \subseteq \sigma(\hat{A})$ and

$$I_n + C(\lambda I_m - A)^{-1}B = I_n + \hat{C}(\lambda I_p - \hat{A})^{-1}\hat{B} \quad \text{for all } \lambda \in \rho(\hat{A}).$$

Therefore, if $(A, B, C; m, n)$ is a node for W, then any dilation $(\hat{A}, \hat{B}, \hat{C}; p, n)$ of the node $(A, B, C; m, n)$ is also a node for W.

3. The two machine flow shop problem

In this section we describe the Two Machine Flow Shop Problem (2MFSP) and some properties of the optimal schedules of instances of 2MFSP. In an instance of 2MFSP there are k jobs that have to be processed by two machines. Each job consists of two operations. The first and the second operation of job j are called O_j^1 and O_j^2 respectively. The first operation O_j^1 must be processed on the first machine, and the second operation O_j^2 must be processed on the second machine. Each machine can be processing at most one operation at the same time. Furthermore, processing O_j^2 on the second machine cannot start until processing O_j^1 on the first machine has been completed.

The processing times of all operations are given and fixed. The processing time of O_j^1 is denoted by s_j and the processing time of O_j^2 is denoted by t_j. Hence an instance J of 2MFSP consists of k tuples (s_j, t_j) specifying the processing times of the operations. Throughout this paper we assume that all processing times are non-negative integers. This is not a serious restriction. What it amounts to is that the processing times are rationals and that the time unit is chosen appropriately. Furthermore, in order to avoid trivialities, we also assume that for each job j either s_j or t_j is non-zero.

If we have a feasible schedule (that is, a schedule satisfying the specified rules), then the length of the time interval required to carry out all jobs is called the *makespan* of the schedule. Now the objective is to find a feasible schedule with *minimum makespan*. The minimum makespan of an instance J is denoted by $\mu(J)$. In the job scheduling literature the makespan is sometimes called the *maximum completion time*. In that case, the minimally obtainable maximum completion time of an instance J is denoted by $C_{\max}(J)$.

It is well-known that each instance of 2MFSP has an optimal *non-preemptive* schedule (cf. [1]). That is, the optimal schedule has the additional property that, once a machine has started processing an operation, it does not start processing another operation until the first operation has been completed. It is also well-known that each instance of 2MFSP has an optimal *permutation* schedule. A schedule is a permutation schedule if it is non-preemptive and for all $i \neq j$ the operations O_i^2 and O_j^2 are processed in the same order as the operations O_i^1 and O_j^1.

These properties of 2MFSP can be proved in a straightforward way by exchange arguments and by using the fact that, given a feasible schedule, an operation on the first machine can be pushed backward in time without violating the predecessor constraints. Similarly, an operation on the second machine can be pushed forward in time without violating the predecessor constraints.

An optimal permutation schedule for an instance of 2MFSP with k jobs can be obtained by the application of *Johnson's Rule* (cf. [1,14]). With Johnson's Rule an optimal permutation schedule is constructed as follows:

- Define the sets V_1 and V_2 by: $V_1 = \{j \mid s_j < t_j\}$ and $V_2 = \{j \mid s_j \geqslant t_j\}$.
- Put the jobs in V_1 in order of increasing s_j, and put the jobs in V_2 in order of decreasing t_j.
- Process the jobs in V_1 first, and process the jobs in V_2 thereafter.

Sorting the jobs in the sets V_1 and V_2 can be accomplished in $\mathcal{O}(k \log k)$ time. Thus the running time of Johnson's Rule is $\mathcal{O}(k \log k)$. Therefore 2MFSP belongs to the class of easy problems that can be solved in polynomial time (cf. [12]).

4. The connection: Description of the results

In this section we describe the connection between quasicomplete factorization of companion based matrix functions and 2MFSP. Recall that a companion based matrix function W admits a minimal realization $W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B$ with A and $A^\times = A - BC$ of first companion type. First, we indicate how an instance J of 2MFSP can be associated with a companion based matrix function W. Thereafter, we describe how the associated instance of 2MFSP can be used to determine the number of elementary factors in a quasicomplete factorization of the companion based matrix function W.

To that end, let W be a companion based $n \times n$ matrix function and let J be an instance of 2MFSP with k jobs (s_j, t_j) where, for $j = 1, \ldots, k$, either s_j or t_j is positive. We say that J is *associated* with W if the pole polynomial p_W and the zero polynomial $p_W^\times$ of W can be written in the form

$$p_W(\lambda) = (\lambda - \beta_1)^{t_1}(\lambda - \beta_2)^{t_2} \cdots (\lambda - \beta_k)^{t_k},$$

$$p_W^\times(\lambda) = (\lambda - \beta_1)^{s_1}(\lambda - \beta_2)^{s_2} \cdots (\lambda - \beta_k)^{s_k},$$

where each β_j is a pole of W, a zero of W (i.e. a pole of W^{-1}), or both $(j = 1, \ldots, k)$. If β_j is a pole and not a zero of W, then $s_j = 0$ and $t_j > 0$. If

β_j is a zero and not a pole of W, then $s_j > 0$ and $t_j = 0$. If β_j is both a pole and a zero of W, then $s_j > 0$ and $t_j > 0$. Note that $\sum_{j=1}^{k} s_j = \sum_{j=1}^{k} t_j = \delta(W)$.

It is obvious that for a given companion based matrix function W there exists an instance J of 2MFSP such that J is associated with W. This instance of 2MFSP is unique up to the ordering of the jobs.

Conversely, if J is an instance of 2MFSP with k jobs as in the preceding paragraph and satisfying $\sum_{j=1}^{k} s_j = \sum_{j=1}^{k} t_j$, then there do exist companion based matrix functions W such that J is associated with W. The latter can be seen as follows. First, choose k different complex numbers $\beta_1, \ldots, \beta_k$ in an arbitrary way. Next, introduce the polynomials $p(\lambda) = (\lambda - \beta_1)^{t_1}(\lambda - \beta_2)^{t_2} \cdots (\lambda - \beta_k)^{t_k}$ and $q(\lambda) = (\lambda - \beta_1)^{s_1}(\lambda - \beta_2)^{s_2} \cdots (\lambda - \beta_k)^{s_k}$. Finally, define the rational matrix function W by

$$W(\lambda) = \begin{bmatrix} 1 & 1/p(\lambda) \\ 0 & q(\lambda)/p(\lambda) \end{bmatrix}.$$

It is not difficult to see that $p_W = p$ and $p_W^{\times} = q$. Furthermore, W is a companion based matrix function (cf. [6]). Therefore, J is associated with W. Also, if R is any invertible 2×2 matrix, then J is associated with $R^{-1}WR$ as well. A similar construction as described here to produce a 2×2 companion based matrix function W such that J is associated with W can be used to find an appropriate $n \times n$ companion based matrix function W. For more details, cf. [5].

Furthermore, if J is an instance of 2MFSP that does not satisfy the condition $\sum_{j=1}^{k} s_j = \sum_{j=1}^{k} t_j$, then this condition can be met by the addition of at least one appropriate dummy job for which one of the processing times equals $|\sum_{j=1}^{k} s_j - \sum_{j=1}^{k} t_j|$, and for which the other processing time equals zero. In this way one obtains an instance J' of 2MFSP that satisfies the desired condition and that is essentially the same as J. In particular, $\mu(J) = \mu(J')$.

Hence, if J is an instance of 2MFSP, then there exist several companion based matrix functions W such that J is associated with W. However, all these functions have basically the same properties with respect to quasicomplete factorization, as will be demonstrated later. So, from the point of view of quasicomplete factorization, these functions can be identified with each other. In this sense, we have uniqueness here as well.

Now we are ready to describe the connection between quasicomplete factorization of companion based matrix functions and 2MFSP. The details are given in Theorem 4.1, which presents the main result of this paper. The theorem is proved in the next section.

Theorem 4.1. *Let W be a companion based rational matrix function and let J be the associated instance of 2MFSP. Then the following connection between the McMillan degree $\delta(W)$, the number $\rho(W)$ of elementary factors in a quasicomplete factorization of W and the minimum makespan $\mu(J)$ of J holds:*

$\rho(W) = \max\{\mu(J) - 1, \delta(W)\}.$

Note that Theorem 4.1 states that either W admits a complete factorization or $\rho(W) = \mu(J) - 1$.

5. The connection: Proof

In this section, we give the proof of Theorem 4.1. The proof has been divided into a number of steps. First, we prove that $\rho(W) \leqslant \max\{\mu(J) - 1, \delta(W)\}$. The reverse inequality is dealt with in the remaining part of this section. We start with Lemmas 5.1 and 5.2 which are used in the first part of the proof of Theorem 4.1. In the first lemma, e_m denotes the mth unit vector in $\mathbb{C}^m$.

Lemma 5.1. *Let $(A,B,C;\ m,n)$ be a node with a controllable pair (A,B) and assume that A and $A^\times = A - BC$ are first companion $m \times m$ matrices. Then there exists an invertible $m \times m$ matrix T such that $AT = TA$, $A^\times T = TA^\times$ and $Te_m \in \operatorname{Ran} B$.*

Proof. The case $A \neq A^\times$ is trivial. Indeed, in this case $A - A^\times = BC \neq O_m$. Thus, $e_m \in \operatorname{Ran}(BC) \subset \operatorname{Ran} B$. Therefore, we can take $T = I_m$ now.

Next, we consider the case $A = A^\times$. The condition $AT = TA$ implies that $T = Q(A)$ is a polynomial in A. It is known (see [18]) that $Q(A)$ is invertible if and only if $Q(\lambda)$ and $P_A(\lambda) = \det(\lambda I_m - A)$ do not have any zeros in common. In addition, we require $Te_m = Q(A)e_m \in \operatorname{Ran} B$.

Write $\operatorname{rank}(B) = r$ and let $x_1, \dots, x_r$ be a basis in Ran B. Since A is a first companion matrix, the mapping $(p_0, \dots, p_{m-1}) \mapsto \sum_{j=0}^{m-1} p_j A^j e_m$ is invertible. Therefore, there exist unique polynomials $P_1, \dots, P_r$ with $P_i(A)e_m = x_i$ for $i = 1, \dots, r$.

We will show that the greatest common divisor $\gcd(P_1, \dots, P_r, P_A) = 1$. Indeed, assume, by contradiction, that there exists $\alpha \in \mathbb{C}$ such that $P_i(\alpha) = 0$ for $i = 1, \dots, r$ and that $P_A(\alpha) = 0$ as well. $P_i(\alpha) = 0$ implies that $x_i = P_i(A)e_m \in \operatorname{Ran}(A - \alpha I_m)$ for $i = 1, \dots, r$. Consequently, $\operatorname{Ran} B = \operatorname{span}\{x_1, \dots, x_r\} \subset \operatorname{Ran}(A - \alpha I_m)$. This implies that

$$\operatorname{Ran}(A - \alpha I_m \quad B) \subset \operatorname{Ran}(A - \alpha I_m).$$

Further, $P_A(\alpha) = 0$ means that α is an eigenvalue for A, so $\operatorname{Ran}(A - \alpha I_m) \neq \mathbb{C}^m$. On the other hand, the controllability of the pair (A, B) is equivalent to

$$\operatorname{rank}(A - \lambda I_m \quad B) = m$$

for all $\lambda \in \mathbb{C}$ (see [4]). A contradiction has been obtained, and hence $\gcd(P_1, \dots, P_r, P_A) = 1$. Now it is not difficult to see that there exist complex numbers $\beta_1, \dots, \beta_r$ such that $Q = \sum_{i=1}^r \beta_i P_i$ satisfies $\gcd(Q, P_A) = 1$. Define

$T = Q(A)$. Then $AT = TA$ holds true, and $\gcd(Q, P_A) = 1$ implies that T is invertible. Finally,

$$Te_m = Q(A)e_m = \sum_{i=1}^{r} \beta_i P_i(A)e_m = \sum_{i=1}^{r} \beta_i x_i \in \operatorname{Ran} B.$$

This proves the lemma. □

Lemma 5.2. *Let W be a companion based rational matrix function and let J be the associated instance of 2MFSP. Furthermore, let $W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B$ be a minimal realization of W with A and $A^\times$ first companion matrices. Then the minimum makespan $\mu(J)$ of the instance J coincides with the minimal integer value $\mu \geqslant m$ for which there exist spectral vectors $(\alpha_1, \ldots, \alpha_m)^{\mathrm{T}}$ and $(\alpha_1^\times, \ldots, \alpha_m^\times)^{\mathrm{T}}$ for A and $A^\times$ such that*

$$\alpha_i \neq \alpha_j^\times, \quad \text{for } i + j \leqslant 2m - \mu + 1.$$

If the ordering condition in the lemma can be achieved for $\mu < m$, then the minimum makespan equals m. Observe that this obvious instance is included in the statement of the lemma. Indeed, in such a case, the ordering condition can also be achieved for $\mu = m$.

Proof of Lemma 5.2. Let $\mu \geqslant m$ be an integer such that the vectors $(\alpha_1, \ldots, \alpha_m)^{\mathrm{T}}$ and $(\alpha_1^\times, \ldots, \alpha_m^\times)^{\mathrm{T}}$ satisfy $\alpha_i \neq \alpha_j^\times$ whenever $i + j \leqslant 2m - \mu + 1$. We will now produce a feasible schedule for J. To that end, the jobs are first labelled by the α's and $\alpha^\times$'s. Now the jobs can be processed in the following schedule, which is not necessarily a permutation schedule. Job α_i is processed on machine 1 during the time interval $[m - i, m - i + 1)$ and job $\alpha_j^\times$ is processed on machine 2 during the time interval $[\mu - m + j - 1, \mu - m + j)$. Job $\alpha_j^\times$ is processed not after job α_i, if $\mu - m + j \leqslant m - i + 1$. In that case, $i + j \leqslant 2m - \mu + 1$ holds true, so $\alpha_i \neq \alpha_j^\times$. It follows that the proposed schedule is feasible and that the time interval during which processing on the machines takes place is given by $[0, \mu)$. In other words, the makespan of this schedule equals μ.

Conversely, let a feasible schedule for J be given with makespan $\mu \geqslant m$. We may assume that the jobs on the first machine are processed in the time interval $[0, m)$, and that the jobs on the second machine are processed during the time interval $[\mu - m, \mu)$. Let α_i be the label of the job processed on the first machine during the time interval $[m - i, m - i + 1)$ and let $\alpha_j^\times$ be the label of the job processed on the second machine during the time interval $[\mu - m + j - 1, \mu - m + j)$. The feasibility of the schedule implies that $\alpha_i \neq \alpha_j^\times$ whenever $\mu - m + j \leqslant m - i + 1$. In other words, $i + j \leqslant 2m - \mu + 1$.

By construction, we have shown that the set of integers $\mu \geqslant m$ representing the makespan of a feasible schedule of the instance J and the set of integers $\mu \geqslant m$ satisfying the ordering condition on the spectral vectors of W are identical. In particular, the conclusion of the lemma follows. □

Proof of Theorem 4.1: First Part. In this first part of the proof of Theorem 4.1, we show that $\rho(W) \leqslant \max\{\mu(J)-1, \delta(W)\}$. To that end, let $W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B$ be a minimal realization for W such that A and $A^\times$ are first companion matrices. Now the proof goes along the following lines: We assume that the first companion matrices A and $A^\times$ have spectral vectors $(\alpha_1, \dots, \alpha_m)^\mathrm{T}$ and $(\alpha_1^\times, \dots, \alpha_m^\times)^\mathrm{T}$ respectively, which satisfy the ordering condition

$$\alpha_i \neq \alpha_j^\times, \quad \text{for } i+j \leqslant 2m-\rho, \tag{4}$$

where the integer $m \leqslant \rho \leqslant 2m-1$ is taken as small as possible. We then construct a $(0, m, \rho-m)$-dilation $(\hat{A}, \hat{B}, \hat{C}; \rho, n)$ of the node $(A, B, C; m, n)$ such that the pair $\hat{A}$, $\hat{A}^\times$ admits simultaneous reduction to complementary triangular forms. By Proposition 2.1, this proves that $\rho(W) \leqslant \rho$. By the minimality of $\rho \geqslant m$, Lemma 5.2 gives that $\rho + 1$ coincides with the minimum makespan $\mu(J)$ of the associated instance J of 2MFSP. As a consequence, $\rho(W) \leqslant \rho = \mu(J) - 1 \leqslant \max\{\mu(J)-1, \delta(W)\}$.

Let $\{\gamma_1, \dots, \gamma_m\}$ be a set of complex numbers, disjoint from the spectra $\sigma(A)$ and $\sigma(A^\times)$. By assumption, the pair of matrices (A, B) satisfies the controllability condition

$$\operatorname{Ran} B + \operatorname{Ran}(AB) + \cdots + \operatorname{Ran}(A^{m-1}B) = \mathbb{C}^m.$$

The pole assignment theorem (see [14]) then gives that there exists an $n \times m$ matrix K such that $A + BK$ has spectrum

$$\sigma(A+BK) = \{\gamma_1, \dots, \gamma_m\}.$$

As a matter of fact, we may assume that $A + BK$ is a first companion matrix. Indeed, by Lemma 5.1, there exists an invertible matrix T with $T^{-1}AT = A$, $T^{-1}A^\times T = A^\times$ and $e_m \in \operatorname{Ran}(T^{-1}B)$. If necessary, we replace the matrix B by $T^{-1}B$ and the matrix C by CT. For that reason, we may assume that $e_m \in \operatorname{Ran} B$. Consequently, there exists a vector $v \in \mathbb{C}^n$ such that $Bv = e_m$. If we write

$$\prod_{j=1}^{m}(\lambda - \alpha_j) = \lambda^m + \sum_{j=0}^{m-1} a_j \lambda^j, \qquad \prod_{j=1}^{m}(\lambda - \gamma_j) = \lambda^m + \sum_{j=0}^{m-1} c_j \lambda^j,$$

then the $n \times m$ matrix

$$K = \begin{pmatrix} (a_0 - c_0)v & (a_1 - c_1)v & \cdots & (a_{m-1} - c_{m-1})v \end{pmatrix}$$

yields the first companion matrix $A + BK$ with eigenvalues $\gamma_1, \dots, \gamma_m$. In the proof, we will only use the first $\rho - m$ eigenvalues $\gamma_1, \dots, \gamma_{\rho-m}$ of the matrix $A + BK$. Furthermore, it is not difficult to see that the $m \times (\rho - m)$ matrix

$$X = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{\rho-m} \\ \vdots & \vdots & & \vdots \\ \gamma_1^{m-1} & \gamma_2^{m-1} & \cdots & \gamma_{\rho-m}^{m-1} \end{pmatrix}$$

and the diagonal $(\rho - m) \times (\rho - m)$ matrix G with diagonal vector $(\gamma_1, \ldots, \gamma_{\rho-m})^{\mathrm{T}}$ satisfy the equation $(A + BK)X = XG$. Put $F = KX$ to obtain $XG - AX = BF$.

Consider the $(0, m, \rho - m)$-dilation $(\hat{A}, \hat{B}, \hat{C}; \rho, n)$ of the node $(A, B, C; m, n)$ with

$$\hat{A} = \begin{pmatrix} A & BF \\ O & G \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} B \\ O \end{pmatrix}, \qquad \hat{C} = (\,C \quad F\,).$$

In order to prove $\rho(W) \leqslant \rho$, it is sufficient now to prove that the pair $\hat{A}$, $\hat{A}^{\times} = \hat{A} - \hat{B}\hat{C}$ admits simultaneous reduction to complementary triangular forms. To that end, we will construct matching maximal invariant nests of subspaces for $\hat{A}$ and $\hat{A}^{\times}$ (see the introduction to this paper and Section 3.1 in [21] for more background information).

First of all, let V and $V^{\times}$ be the invertible $m \times m$ generalized Vandermonde matrices such that $V^{-1}AV$ is upper triangular with diagonal $(\alpha_1, \ldots, \alpha_m)^{\mathrm{T}}$ and $(V^{\times})^{-1}A^{\times}V^{\times}$ is upper triangular with diagonal $(\alpha_1^{\times}, \ldots, \alpha_m^{\times})^{\mathrm{T}}$. For details on (generalized) eigenvectors of first companion matrices, see Exercise 22 on p. 69 in [18].

Fix $k \in \{0, \ldots, m\}$, and let $M_k = \mathrm{Ran}(VE_m[k])$ be the k-dimensional invariant subspace for A, corresponding to the spectral vector $(\alpha_1, \ldots, \alpha_k)^{\mathrm{T}}$. In other words, M_k is the span of the first k columns of V. In addition, let $M_k^{\times} = \mathrm{Ran}(V^{\times}E_m[k])$ be the k-dimensional invariant subspace for $A^{\times}$, corresponding to the spectral vector $(\alpha_1^{\times}, \ldots, \alpha_k^{\times})^{\mathrm{T}}$. We also define the invertible $\rho \times \rho$ matrix

$$\hat{X} = \begin{pmatrix} I_m & X \\ O & I_{\rho-m} \end{pmatrix}.$$

Next, we define the subspaces

$$\hat{M}_k = \hat{X}\begin{pmatrix} (0) \\ \mathrm{Ran}\ E_{\rho-m}[k] \end{pmatrix}, \quad k = 0, \ldots, \rho - m,$$

$$\hat{M}_k = \hat{X}\begin{pmatrix} M_{k+m-\rho} \\ \mathbb{C}^{\rho-m} \end{pmatrix}, \quad k = \rho - m, \ldots, \rho,$$

$$\hat{M}_k^{\times} = \begin{pmatrix} (0) \\ \mathrm{Ran}\ E_{\rho-m}[k] \end{pmatrix}, \quad k = 0, \ldots, \rho - m,$$

$$\hat{M}_k^\times = \begin{pmatrix} M_{k+m-\rho}^\times \\ \mathbb{C}^{\rho-m} \end{pmatrix}, \quad k = \rho - m, \ldots, \rho.$$

The nest $\{\hat{M}_k\}_{k=0}^{\rho}$ consists of invariant subspaces for $\hat{A}$, since

$$\hat{X}^{-1}\hat{A}\hat{X} = \begin{pmatrix} I_m & -X \\ O & I_{\rho-m} \end{pmatrix}\begin{pmatrix} A & BF \\ O & G \end{pmatrix}\begin{pmatrix} I_m & X \\ O & I_{\rho-m} \end{pmatrix} = \begin{pmatrix} A & O \\ O & G \end{pmatrix},$$

where we have used the fact that $XG - AX = BF$. Further, $\{\hat{M}_k^\times\}_{k=0}^{\rho}$ is a maximal nest of invariant subspaces for $\hat{A}^\times$, since

$$\hat{A}^\times = \begin{pmatrix} A^\times & O \\ O & G \end{pmatrix}.$$

We will now verify the matching condition

$$\hat{M}_k \oplus \hat{M}_{\rho-k}^\times = \begin{pmatrix} \mathbb{C}^m \\ \mathbb{C}^{\rho-m} \end{pmatrix}, \quad \text{for } k = 0, \ldots, \rho.$$

Note that the inequalities $m \leqslant \rho \leqslant 2m - 1$ imply $0 \leqslant \rho - m \leqslant m \leqslant \rho$. If $k \in \{0, \ldots, \rho - m\}$, then

$$\hat{M}_k = \hat{X}\begin{pmatrix} (0) \\ \operatorname{Ran} E_{\rho-m}[k] \end{pmatrix}, \qquad \hat{M}_{\rho-k}^\times = \begin{pmatrix} \operatorname{Ran}(V^\times E_m[m-k]) \\ \mathbb{C}^{\rho-m} \end{pmatrix}.$$

An equivalent statement to $\hat{M}_k \cap \hat{M}_{\rho-k}^\times = (0)$ is $\operatorname{Ran}(XE_{\rho-m}[k]) \cap \operatorname{Ran}(V^\times E_m\ [m-k]) = (0)$. The latter holds, since $\sigma(G) \cap \sigma(A^\times) = \emptyset$. If $k \in \{\rho - m, \ldots, m\}$, then

$$\hat{M}_k = \hat{X}\begin{pmatrix} \operatorname{Ran}(VE_m[k+m-\rho]) \\ \mathbb{C}^{\rho-m} \end{pmatrix}, \qquad \hat{M}_{\rho-k}^\times = \begin{pmatrix} \operatorname{Ran}(V^\times E_m[m-k]) \\ \mathbb{C}^{\rho-m} \end{pmatrix}.$$

These spaces match if and only if $\operatorname{Ran}(VE_m[k+m-\rho]) \oplus \operatorname{Ran}(V^\times E_m[m-k]) \oplus \operatorname{Ran} X = \mathbb{C}^m$. This is indeed the case: First of all, $\operatorname{Ran}(VE_m[k+m-\rho]) \cap \operatorname{Ran}(V^\times E_m[m-k]) = (0)$ follows from the ordering condition (4) on the spectral vectors. Secondly, $\sigma(G) \cap \sigma(A) = \sigma(G) \cap \sigma(A^\times) = \emptyset$ implies $\{\operatorname{Ran}(VE_m[k+m-\rho]) \oplus \operatorname{Ran}(V^\times E_m[m-k])\} \cap \operatorname{Ran} X = (0)$.

Finally, if $k \in \{m, \ldots, \rho\}$, then

$$\hat{M}_k = \hat{X}\begin{pmatrix} \operatorname{Ran}(VE_m[k+m-\rho]) \\ \mathbb{C}^{\rho-m} \end{pmatrix}, \qquad \hat{M}_{\rho-k}^\times = \begin{pmatrix} (0) \\ \operatorname{Ran}(E_{\rho-m}[\rho-k]) \end{pmatrix}.$$

The two spaces match if and only if $\operatorname{Ran}(XE_{\rho-m}[\rho-k]) \cap \operatorname{Ran}(VE_m[k+m-\rho]) = (0)$. This is indeed the case, since $\sigma(G) \cap \sigma(A) = \emptyset$.

We have shown that the pair $\hat{A}$, $\hat{A}^\times$ admits simultaneous reduction to complementary triangular forms. This concludes the first part of the proof of Theorem 4.1. □

The remaining part of this section is devoted to the proof of the reverse inequality $\rho(W) \geqslant \max\{\mu(J)-1, \delta(W)\}$. Recall that $\rho(W) \geqslant \delta(W)$ is obviously true for any rational matrix function. Therefore, it remains to prove that $\rho(W) \geqslant \mu(J)-1$.

The main idea of the proof is as follows. To verify that $\rho(W) \geqslant \mu(J)-1$ or, alternatively, that $\mu(J) \leqslant \rho(W)+1$, we show that whenever W admits a factorization into ρ elementary factors $W = W_1 \cdots W_\rho$, we can find a feasible schedule of J with makespan $\mu = \rho+1$. Thus, the minimum makespan of J satisfies $\mu(J) \leqslant \rho+1$. Minimizing the right-hand side of this inequality leads to $\mu(J) \leqslant \rho(W)+1$. We will give the argument in detail by means of three lemmas which are put together in a short proof at the end of this section.

Lemma 5.3. *Let m, ρ and κ be integers satisfying $m \leqslant \rho \leqslant 2m-1$ and $0 \leqslant \kappa \leqslant \rho - m$. Furthermore, let $(A,B,C;m,n)$ be a minimal node for the rational matrix function W, and let $(\hat{A},\hat{B},\hat{C};\rho,n)$ be a $(\kappa,m,\rho-m-\kappa)$-dilation of this minimal node such that the pair $\hat{A}$, $\hat{A}^\times$ admits simultaneous reduction to complementary triangular forms. Then there exist maximal invariant nests of subspaces $\{M_k\}_{k=0}^m$ and $\{M_k^\times\}_{k=0}^m$ of A and $A^\times = A - BC$ respectively, such that for all integers $0 \leqslant k, l \leqslant m$, either $\dim(M_k \cap M_l^\times) \leqslant \kappa$ or $\dim(M_k + M_l^\times) \geqslant 2m - \rho + \kappa$.*

Proof. In this paper, we present a concise proof of this lemma. Some details (see the claims below) are left to the reader here and can be found in [8].

Claim 1: With respect to the decomposition $\mathbb{C}^\rho = \mathbb{C}^\kappa \oplus \mathbb{C}^m \oplus \mathbb{C}^{\rho-m-\kappa}$, consider the partitioned $\rho \times \rho$ matrix

$$\hat{A} = \begin{pmatrix} A_1 & A_{12} & A_{13} \\ O & A & A_{23} \\ O & O & A_3 \end{pmatrix},$$

and define $\lambda(\hat{M}) = (\hat{M} + \mathbb{C}^\kappa) \cap \mathbb{C}^m$ for a subspace $\hat{M} \subset \mathbb{C}^\rho$. Then the following statements hold.

1. If $\hat{M}$ is an invariant subspace for $\hat{A}$, then $\lambda(\hat{M})$ is an invariant subspace for A.
2. If $\hat{M} \subset \hat{N}$ are subspaces, then $\lambda(\hat{M}) \subset \lambda(\hat{N})$, and $\dim(\lambda(\hat{N})/\lambda(\hat{M})) \leqslant \dim(\hat{N}/\hat{M})$.
3. If $\{\hat{M}_k\}_{k=0}^\rho$ is a maximal invariant nest of subspaces for $\hat{A}$, then $\{\lambda(\hat{M}_k)\}_{k=0}^\rho$ contains a maximal invariant nest of subspaces for A.

Claim 2: Let $\hat{M} \subset \mathbb{C}^\rho$, and let the decomposition $\mathbb{C}^\rho = \mathbb{C}^\kappa \oplus \mathbb{C}^m \oplus \mathbb{C}^{\rho-m-\kappa}$ be given. Define $\lambda(\hat{M}) = (\hat{M} + \mathbb{C}^\kappa) \cap \mathbb{C}^m$. Let P be the projection onto $\mathbb{C}^{\rho-m-\kappa}$ along $\mathbb{C}^\kappa \oplus \mathbb{C}^m$. Then

$$\dim\,(\hat{M} \cap \mathbb{C}^\kappa) + \dim\,\lambda(\hat{M}) + \dim\,P(\hat{M}) = \dim\,\hat{M}.$$

Claim 3: Given the aforementioned decomposition of $\mathbb{C}^\rho$ and a subspace $\hat{M} \subset \mathbb{C}^\rho$ of dimension $\dim \hat{M} = \hat{m}$, one may construct a $\rho \times \hat{m}$ matrix

$$\Lambda(\hat{M}) = \begin{pmatrix} V_1 & V_{12} & V_{13} \\ O & V & V_{23} \\ O & O & V_3 \end{pmatrix},$$

such that $\operatorname{Ran} V_1 = \hat{M} \cap \mathbb{C}^\kappa, \quad \operatorname{Ran} V = \lambda(\hat{M}), \quad \operatorname{Ran} V_3 = P(\hat{M})$.

Using the three claims, we finish the proof of the lemma. We assume that the pair of $\rho \times \rho$ matrices $\hat{A}, \hat{A}^\times$ admits simultaneous reduction to complementary triangular forms, and let the matching maximal invariant nests for $\hat{A}$ and $\hat{A}^\times$ be given by $\{\hat{M}_k\}_{k=0}^\rho$ and $\{\hat{M}_k^\times\}_{k=0}^\rho$, respectively. Claim 1 implies that $\{\lambda(\hat{M}_k)\}_{k=0}^\rho$ and $\{\lambda(\hat{M}_k^\times)\}_{k=0}^\rho$ respectively contain maximal invariant nests of subspaces for A and $A^\times$. It remains to prove the dimension estimates as stated in the lemma.

Fix $0 \leqslant k, l \leqslant m$, and, as in Claim 3, consider the matrices

$$\Lambda(\hat{M}_k) = \begin{pmatrix} V_1 & V_{12} & V_{13} \\ O & V & V_{23} \\ O & O & V_3 \end{pmatrix}, \qquad \Lambda(\hat{M}_l^\times) = \begin{pmatrix} V_1^\times & V_{12}^\times & V_{13}^\times \\ O & V^\times & V_{23}^\times \\ O & O & V_3^\times \end{pmatrix}.$$

We distinguish two cases.

Case 1: $k + l \geqslant \rho$, so $\hat{M}_k + \hat{M}_l^\times = \mathbb{C}^\rho$. In this case,

$$\begin{aligned} \rho = \operatorname{rank} & \begin{pmatrix} V_1 & V_1^\times & V_{12} & V_{12}^\times & V_{13} & V_{13}^\times \\ O & O & V & V^\times & V_{23} & V_{23}^\times \\ O & O & O & O & V_3 & V_3^\times \end{pmatrix} \\ & \leqslant \kappa + 2(\rho - m - \kappa) + \operatorname{rank}(\, V \; V^\times). \end{aligned}$$

This implies $\dim(\lambda(\hat{M}_k) + \lambda(\hat{M}_l^\times)) = \operatorname{rank}(V \; V^\times\,) \geqslant \rho - \kappa - 2(\rho - m - \kappa) = 2m - \rho + \kappa$.

Case 2: $k + l \leqslant \rho$, so $\hat{M}_k \cap \hat{M}_l^\times = (0)$. In this case,

$$\dim \lambda(\hat{M}_k) + \dim \lambda(\hat{M}_l^\times) = \operatorname{rank}\begin{pmatrix} V_{12} & V_{12}^\times \\ V & V^\times \end{pmatrix} \leqslant \kappa + \operatorname{rank}(\, V \; V^\times).$$

Therefore, $\dim(\lambda(\hat{M}_k) \cap \lambda(\hat{M}_l^\times)) = \dim \lambda(\hat{M}_k) + \dim \lambda(\hat{M}_l^\times) - \dim(\lambda(\hat{M}_k) + \lambda(\hat{M}_l^\times)) = \dim \lambda(\hat{M}_k) + \dim \lambda(\hat{M}_l^\times) - \operatorname{rank}(\, V \; V^\times) \leqslant \kappa$. This proves the lemma. $\square$

Lemma 5.4. *Let m, ρ and κ be integers satisfying $m \leqslant \rho \leqslant 2m - 1$ and $0 \leqslant \kappa \leqslant \rho - m$. Furthermore, let $\{M_k\}_{k=0}^m$ and $\{M_k^\times\}_{k=0}^m$ be maximal nests of subspaces such that either* $\dim(M_k \cap M_l^\times) \leqslant \kappa$ *or* $\dim(M_k + M_l^\times) \geqslant 2m - \rho + \kappa$ *for all integers $0 \leqslant k, l \leqslant m$. Then* $\dim(M_k \cap M_l^\times) \leqslant \kappa$ *whenever $k + l \leqslant 2m - \rho + 2\kappa$.*

Proof. Assume that $k + l \leqslant 2m - \rho + 2\kappa$. If dim $(M_k \cap M_l^\times) \leqslant \kappa$, then we are ready. On the other hand, if dim $(M_k + M_l^\times) \geqslant 2m - \rho + \kappa$, then dim $(M_k \cap M_l \times) \leqslant k + l - (2m - \rho + \kappa) \leqslant \kappa$. This proves the lemma. □

Lemma 5.5. *Let m, ρ and κ be integers satisfying* $m \leqslant \rho \leqslant 2m - 1$ *and* $0 \leqslant \kappa \leqslant \rho - m$. *Assume that the following condition is satisfied:*

(†) *The vectors* $(\alpha_1, \ldots, \alpha_l)^{\mathrm{T}}$ *and* $(\alpha_1^\times, \ldots, \alpha_{2m-\rho+2\kappa-l}^\times)^{\mathrm{T}}$ *have at most κ elements in common, counted according to multiplicity, for* $l = \kappa + 1, \ldots, 2m - \rho + \kappa - 1$.

Then there exist permutations π and $\pi^\times$ *of the set* $\{1, \ldots, m\}$ *such that*

$$\alpha_{\pi(i)} \neq \alpha_{\pi^\times(j)}^\times, \qquad \textit{for } i + j \leqslant 2m - \rho. \tag{5}$$

The multiplicity notion in (†) is probably intuitively clear but will be explained in detail at the beginning of the proof of the lemma.

Proof of Lemma 5.5. Assume that the vectors $(\alpha_1, \ldots, \alpha_m)^{\mathrm{T}}$ and $(\alpha_1^\times, \ldots, \alpha_m^\times)^{\mathrm{T}}$ satisfy condition (†). For $l = \kappa + 1, \ldots, 2m - \rho + \kappa - 1$ and $\alpha \in \mathbb{C}$, define the integers

$$n_\alpha(l) = \#\{j \mid 1 \leqslant j \leqslant l,\ \alpha_j = \alpha\},$$
$$n_\alpha^\times(l) = \#\{j \mid 1 \leqslant j \leqslant l,\ \alpha_j^\times = \alpha\}.$$

Thus the integer $n_\alpha(l)$, for instance, counts the number of times the complex number $\alpha \in \mathbb{C}$ appears as an element in the vector $(\alpha_1, \ldots, \alpha_l)^{\mathrm{T}}$. Further, write

$$n(l) = \sum_\alpha \min\{n_\alpha(l), n_\alpha^\times(2m - \rho + 2\kappa - l)\},$$

where $\sum_\alpha$ denotes the finite sum over the complex numbers $\alpha \in \{\alpha_k\}_{k=1}^m \cap \{\alpha_k^\times\}_{k=1}^m$. In terms of these integers $n(l)$, condition (†) can be rewritten as

$$n(l) \leqslant \kappa, \quad \text{for } l = \kappa + 1, \ldots, 2m - \rho + \kappa - 1.$$

Now let $\kappa + 1 \leqslant l \leqslant 2m - \rho + \kappa - 1$. We define the $n(l)$ distinct integers $1 \leqslant t_l(1), \ldots, t_l(n(l)) \leqslant l$ and the $n(l)$ distinct integers $1 \leqslant t_l^\times(1), \ldots, t_l^\times(n(l)) \leqslant 2m - \rho + 2\kappa - l$ such that $\alpha_{t_l(i)} = \alpha_{t_l^\times(i)}^\times$ for $i = 1, \ldots, n(l)$.

We choose the integers $t_l(i)$ and $t_l^\times(i)$ in such a way that they indicate the positions of the $n(l)$ 'leftmost' common elements in the vectors $(\alpha_1, \ldots, \alpha_l)^{\mathrm{T}}$ and $(\alpha_1^\times, \ldots, \alpha_{2m-\rho+2\kappa-l}^\times)^{\mathrm{T}}$. Indeed, we start with identifying the $n(l)$ common elements in the vectors, counted according to multiplicity. Then we search in each of the two separate vectors for these $n(l)$ elements, starting with the low-index positions. To make this procedure more clear, we illustrate it in the following intermezzo.

Example 5.6. The procedure to construct the positions $t_l(i)$ and $t_l^\times(i)$ is illustrated by the following example. Let $l = 9$ and $2m - \rho + 2\kappa - l = 11$. Furthermore, let the vectors $(\alpha_1, \dots, \alpha_l)^{\mathrm{T}}$ and $(\alpha_1^\times, \dots, \alpha_{2m-\rho+2\kappa-l}^\times)^{\mathrm{T}}$ be given by

$$(\alpha, \gamma, \alpha, \alpha, \beta, \beta, \gamma, \beta, \gamma)^{\mathrm{T}}, \ (\delta, \alpha, \alpha, \beta, \delta, \delta, \beta, \alpha, \delta, \alpha, \delta)^{\mathrm{T}}.$$

Then the common elements, counted according to multiplicity, are given by $(\alpha, \alpha, \alpha, \beta, \beta)^{\mathrm{T}}$, so $n(l) = 5$. If we search for these elements in the vectors starting from the left, we find the positions $t_l(1) = 1$, $t_l(2) = 3$, $t_l(3) = 4$, $t_l(4) = 5$, $t_l(5) = 6$ and $t_l^\times(1) = 2$, $t_l^\times(2) = 3$, $t_l^\times(3) = 8$, $t_l^\times(4) = 4$, $t_l^\times(5) = 7$. Note that the integers $t_l(i)$ and $t_l^\times(i)$ should indicate positions of one and the same complex value for $i = 1, \dots, n(l)$, so that in general we cannot expect both $t_l(.)$ and $t_l^\times(.)$ to be increasing sequences.

After this intermezzo, we will prove next that there exist distinct integers $\pi(1), \dots, \pi(2m - \rho - 1)$ and distinct integers $\pi^\times(1), \dots, \pi^\times(2m - \rho - 1)$ such that

$$\alpha_{\pi(i)} \neq \alpha^\times_{\pi^\times(j)}, \quad \text{for } i + j \leqslant 2m - \rho.$$

To that end, we define the sets $\mathcal{O}_l = \{t_l(j) \mid j = 1, \dots, n(l)\}$ and $\mathcal{O}_l^\times = \{t_l^\times(j) \mid j = 1, \dots, n(l)\}$, and we also put $\mathcal{O}_{2m-\rho+\kappa} = \emptyset$ and $\mathcal{O}_\kappa^\times = \emptyset$. We first prove the following claim.

Claim 1. *If* $1 \leqslant t \leqslant l$ *and* $t \notin \mathcal{O}_l$, *then* $t \notin \mathcal{O}_{l+1}$.

Proof of Claim 1. To avoid trivialities, we only consider the case $\kappa + 1 \leqslant l < 2m - \rho + \kappa - 1$. Assume that $1 \leqslant t \leqslant l$ and $t \notin \mathcal{O}_l$. Furthermore, write $\alpha = \alpha_t$.

First, note that $n_\alpha(l) > n_\alpha^\times(2m - \rho + 2\kappa - l)$. Indeed, $t \notin \mathcal{O}_l$, together with the way we constructed the positions $t_l(i)$ and $t_l^\times(i)$, implies that not all α's in the vector $(\alpha_1, \dots, \alpha_l)^{\mathrm{T}}$ have been matched with an α in the vector $(\alpha_1^\times, \dots, \alpha_{2m-\rho+2\kappa-l}^\times)^{\mathrm{T}}$. Therefore, the second vector contains the least number of α's. We further get $n_\alpha(l+1) \geqslant n_\alpha(l) > n_\alpha^\times(2m - \rho + 2\kappa - l) \geqslant n_\alpha^\times(2m - \rho + 2\kappa - l - 1)$. Therefore,

$$\begin{aligned} &\min\{n_\alpha(l), n_\alpha^\times(2m - \rho + 2\kappa - l)\} \\ &\quad \geqslant \min\{n_\alpha(l+1), n_\alpha^\times(2m - \rho + 2\kappa - l - 1)\}. \end{aligned}$$

This implies $\{s \in \mathcal{O}_{l+1} \mid \alpha_s = \alpha\} \subset \{s \in \mathcal{O}_l \mid \alpha_s = \alpha\}$. Thus, if $1 \leqslant t \leqslant l$ and $t \notin \mathcal{O}_l$, then also $t \notin \mathcal{O}_{l+1}$. The claim has been proved. □

In the same fashion, one can prove the following claim.

Claim 2. *If* $1 \leqslant t \leqslant 2m - \rho + 2\kappa - l$ *and* $t \notin \mathcal{O}_l^\times$, *then* $t \notin \mathcal{O}_{l-1}^\times$.

We will now define the permutations π and $\pi^\times$ requested in the lemma. First, we choose $\pi(1) \in \{1, \dots, \kappa+1\} \setminus \mathcal{O}_{\kappa+1}$. Next, if we have chosen $\pi(1), \dots, \pi(s)$ such that $\pi(j) \in \{1, \dots, \kappa+j\} \setminus \mathcal{O}_{\kappa+j}$ for $j = 1, \dots, s$, then Claim 1 implies that $\{\pi(1), \dots, \pi(s)\}$ and $\mathcal{O}_{\kappa+s+1}$ are disjoint sets. The union of these two sets contains $s + n(\kappa+s+1) \leqslant s+\kappa$ elements, so there exists an integer $\pi(s+1) \in \{1, \dots, \kappa+s+1\}$ which is not in this union. In other words, we can choose $\pi(s+1) \in \{1, \dots, \kappa+s+1\}$ such that $\pi(s+1) \neq \pi(j)$ for $j = 1, \dots, s$ and such that $\pi(s+1) \notin \mathcal{O}_{\kappa+s+1}$. Proceeding in this manner, we obtain $2m-\rho-1$ distinct integers $\pi(1), \dots, \pi(2m-\rho-1)$.

Furthermore, the given inequality $m \leqslant \rho$ implies $2m-\rho-1 \leqslant m$, and the given inequality $\kappa \leqslant \rho - m$ implies $\pi(i) \leqslant \kappa + i \leqslant m$ for $i = 1, \dots, 2m-\rho-1$. Thus, by abuse of notation, we can identify the integers $\pi(i)$ with the values of a permutation π of the set $\{1, \dots, m\}$. The permutation π can be completed in an arbitrary way.

In the same fashion, we can choose distinct integers $\pi^\times(1), \dots, \pi^\times(2m-\rho-1)$ such that $\pi^\times(j) \in \{1, \dots, \kappa+j\} \setminus \mathcal{O}^\times_{2m-\rho+\kappa-j}$ for $j = 1, \dots, 2m-\rho-1$. As before, we identify these integers with the values of a permutation $\pi^\times$ of the set $\{1, \dots, m\}$. The permutation $\pi^\times$ can also be completed in an arbitrary way.

Now we still have to prove the conclusion of the lemma. To that end, let $i + j \leqslant 2m - \rho$. Note that $\pi(i) \notin \mathcal{O}_{\kappa+i}$ and $\pi^\times(j) \notin \mathcal{O}^\times_{2m-\rho+\kappa-j}$. So, by Claim 2 and $\kappa + i \leqslant 2m-\rho+\kappa-j$, we get $\pi^\times(j) \notin \mathcal{O}^\times_{\kappa+i}$. By the definition of $\mathcal{O}_{\kappa+i}$ and $\mathcal{O}^\times_{\kappa+i}$, the latter implies $\alpha_{\pi(i)} \neq \alpha^\times_{\pi^\times(j)}$. Indeed, we have $1 \leqslant \pi^\times(j) \leqslant \kappa + j$. Combining this with $\kappa + j \leqslant 2m-\rho+\kappa-i$, we find $1 \leqslant \pi^\times(j) \leqslant 2m-\rho+\kappa-i$. As a consequence, if $\alpha_{\pi(i)} = \alpha^\times_{\pi^\times(j)}$ were true, then either $\pi(i) \in \mathcal{O}_{\kappa+i}$ or $\pi^\times(j) \in \mathcal{O}^\times_{\kappa+i}$. The lemma has been proved. □

Now we are ready to finish the proof of Theorem 4.1.

Proof of Theorem 4.1: Second Part. Assume that W admits a factorization into ρ elementary factors $W = W_1 \cdots W_\rho$. By Proposition 2.1, there exists a $(\kappa, m, \rho-m-\kappa)$-dilation $(\hat{A}, \hat{B}, \hat{C}; \rho, n)$ of the minimal node $(A, B, C; m, n)$ such that the pair of $\rho \times \rho$ matrices $\hat{A}$, $\hat{A}^\times$ admits simultaneous reduction to complementary triangular forms.

Lemma 5.3 implies that there exist maximal invariant nests $\{M_k\}_{k=0}^m$ and $\{M_k^\times\}_{k=0}^m$ of A and $A^\times$ respectively, such that for all integers $0 \leqslant k, l \leqslant m$, either $\dim\,(M_k \cap M_l^\times) \leqslant \kappa$ or $\dim\,(M_k + M_l^\times) \geqslant 2m - \rho + \kappa$. Furthermore, Lemma 5.4 gives $\dim\,(M_k \cap M_l^\times) \leqslant \kappa$ whenever $k + l \leqslant 2m - \rho + 2\kappa$.

Since A and $A^\times$ are first companion matrices, the latter condition is equivalent to the following ordering condition: There exist spectral vectors $(\alpha_1, \dots, \alpha_m)^{\mathrm{T}}$ and $(\alpha_1^\times, \dots, \alpha_m^\times)^{\mathrm{T}}$ of A and $A^\times$ respectively, such that for $l = \kappa+1, \dots, 2m-\rho+\kappa-1$ the vectors $(\alpha_1, \dots, \alpha_l)^{\mathrm{T}}$ and $(\alpha_1^\times, \dots, \alpha^\times_{2m-\rho+2\kappa-l})^{\mathrm{T}}$ have at most κ elements in common, multiplicities included.

Lemma 5.5 then provides two permutations π and $\pi^\times$ of the set $\{1, \ldots, m\}$ such that $\alpha_{\pi(i)} \neq \alpha^\times_{\pi^\times(j)}$ whenever $i + j \leqslant 2m - \rho = 2m - (\rho + 1) + 1$.

Finally, Lemma 5.2 yields $\mu(J) \leqslant \rho + 1$. Therefore, $\mu(J) \leqslant \rho(W) + 1$. □

The results in this section lead to the interesting observation of Corollary 5, which shows that the estimate $\rho(W) \leqslant 2\delta(W) - 1$ from [21] is sharp, in the sense that for each positive integer m there exists a rational matrix function W of McMillan degree $\delta(W) = m$ with $\rho(W) = 2\delta(W) - 1$. Note that for the proof of this corollary only Lemma 5.3 is required.

Corollary 5.7. *The* 2×2 *companion based matrix function*

$$W(\lambda) = \begin{pmatrix} 1 & 1/\lambda^m \\ 0 & 1 \end{pmatrix}$$

has McMillan degree $\delta(W) = m$ *and satisfies* $\rho(W) = 2\delta(W) - 1$.

We mention the fact that these rational matrix functions are *irreducible* in the sense that they do not admit any minimal factorization into non-trivial factors.

Proof of Corollary 5.7. First of all, in [21] the inequality $\rho(W) \leqslant 2\delta(W) - 1$ is shown to be valid for all rational matrix functions. Next, note that a minimal realization for W is given by $W(\lambda) = I_2 + C(\lambda I_m - A)^{-1}B$, where $A = A^\times = J(0; m)$, the nilpotent upper triangular $m \times m$ Jordan block. Since $J(0; m)$ is a first companion matrix, it follows that W is companion based. If $(\hat{A}, \hat{B}, \hat{C}; \rho, 2)$ is a $(\kappa, m, \rho - m - \kappa)$-dilation of the node $(A, B, C; m, 2)$ for which the pair $\hat{A}$, $\hat{A}^\times$ admits simultaneous reduction to complementary triangular forms, then we know from Lemma 5.3 that there exist maximal invariant nests of subspaces $\{M_k\}_{k=0}^m$ and $\{M_k^\times\}_{k=0}^m$ for A and $A^\times$ respectively, such that for all $0 \leqslant k, l \leqslant m$, either $\dim\,(M_k + M_l^\times) \geqslant 2m - \rho + \kappa$ or $\dim\,(M_k \cap M_l^\times) \leqslant \kappa$.

Since $A = A^\times = J(0; m)$ has a unique maximal nest of invariant subspaces, we get

$$\dim(M_k + M_l^\times) = \max\{k, l\}, \qquad \dim(M_k \cap M_l^\times) = \min\{k, l\}.$$

In case $k = l$, we find $k \geqslant 2m - \rho + \kappa$ or $k \leqslant \kappa$ for all $k = 0, \ldots, m$. These inequalities can be satisfied only if there is no 'wrong gap' between the integers $2m - \rho + \kappa$ and κ. In other words, $2m - \rho + \kappa \leqslant \kappa + 1$, or $2m - 1 \leqslant \rho$. Thus $2m - 1 = 2\delta(W) - 1 \leqslant \rho(W)$. □

To illustrate quasicomplete factorizations for the irreducible rational matrix functions in Corollary 5.7, we shall give explicit factorizations for $m = 2$ and $m = 3$:

$$\begin{pmatrix} 1 & 1/\lambda^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/(\lambda-1) \\ 0 & \lambda/(\lambda-1) \end{pmatrix} \begin{pmatrix} 1 & -1/\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (\lambda-1)/\lambda \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1/\lambda^3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/(\lambda-1) \\ 0 & \lambda/(\lambda-1) \end{pmatrix} \begin{pmatrix} 1 & 2/(2\lambda+1) \\ 0 & 2\lambda/(2\lambda+1) \end{pmatrix} \begin{pmatrix} 1 & -2/\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (\lambda-1)/\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (2\lambda+1)/2\lambda \end{pmatrix}.$$

Although the proof of the main result in [22] provides a more or less constructive method to find quasicomplete factorizations for these rational matrix functions, the above factorizations have been obtained using ad hoc methods.

6. Optimization problems

In this section, we discuss two optimization problems from [6,7] in relation to the main result of this paper, Theorem 4.1.

Let W be a companion based matrix function and let J be the associated instance of 2MFSP as described in Section 4. In this section we are looking for schedules for J that respect a certain deadline D. That is, all jobs should be completed within D time units. Obviously, if $D < \mu(J)$, then this is only possible by relaxing the predecessor constraints of the operations of the jobs. Thus, in this case operation O_j^2 of job j may start already on machine 2 before operation O_j^1 on machine 1 has been completed.

In order to make this more precise, we introduce the following notation. For a given schedule, the starting time of operation O_j^ν is denoted by $S(O_j^\nu)$, and the finishing time of operation O_j^ν is denoted by $F(O_j^\nu)$, where $j = 1, \dots, k$ and $\nu = 1, 2$. Now the *infeasibility* $\mathscr{I}_j$ of job j is defined as

$$\mathscr{I}_j = \max\{F(O_j^1) - S(O_j^2),\ 0\}.$$

The first optimization problem is to find an optimal schedule in the following sense: An optimal schedule respects the deadline D and minimizes $\max\{\mathscr{I}_j \mid j = 1, \dots, k\}$. We write

$$\gamma(J) = \min\ \max\{\mathscr{I}_j \mid j = 1, \dots, k\},$$

where the maximum is taken over the k jobs and the minimum is taken over all possible schedules. Now we give the following result from [6].

Lemma 6.1. *Let J be an instance of 2MFSP and let D be the deadline to be respected. Then $\gamma(J) = \max\{\mu(J) - D, 0\}$.*

In [6] it has also been shown that the same permutation schedule that minimizes $\mu(J)$ also minimizes $\gamma(J)$. In other words, an optimal schedule can be determined by the application of Johnson's Rule (cf. Section 3).

For an $n \times n$ rational matrix function W, the positive integer $\beta(W)$ minimizes the maximal McMillan degree of the factors which appear in a minimal factorization of W. Thus,

$$\beta(W) = \min\ \max\left\{ \delta(W_i) \mid W = W_1 \dots W_p;\ \delta(W) = \sum_{i=1}^{p} \delta(W_i) \right\},$$

where the maximum is taken over the McMillan degrees of the factors of a fixed minimal factorization, while the minimum is taken over all possible minimal factorizations of W. If W does not admit a non-trivial minimal factorization, then $\beta(W) = \delta(W)$. If W admits complete factorization, then $\beta(W) = 1$. In [8], the following result has been shown.

Lemma 6.2. *Let W be a companion based rational matrix function and let J be the associated instance of 2MFSP with deadline $D = \delta(W)$. Then $\beta(W) = \max\{\gamma(J), 1\}$.*

Lemmas 6.1 and 6.2 together with Theorem 4.1 yield the following result.

Theorem 6.3. *Let W be a companion based rational matrix function. Then*

$$\rho(W) - \beta(W) = \delta(W) - 1.$$

Proof. Let J be the associated instance of 2MFSP with deadline $D = \delta(W)$. By the two preceeding lemmas, we get $\beta(W) + \delta(W) - 1 = \max\{\gamma(J) + \delta(W) - 1, \delta(W)\} = \max\{\gamma(J) + D - 1, \delta(W)\} = \max\{\mu(J) - 1, \delta(W)\}$. Now Theorem 4.1 yields $\beta(W) + \delta(W) - 1 = \rho(W)$. This completes the proof of the theorem. □

The conclusion of Theorem 6.3 does not hold for arbitrary rational matrix functions, as the following example shows.

Example 6.4. The rational 3×3 matrix function

$$W(\lambda) = \begin{pmatrix} 1 - 1/2\lambda & (\lambda + 1)/2\lambda^2 & -(2\lambda^2 + \lambda + 1)/2\lambda^3 \\ 1/2\lambda & 1 + (\lambda - 1)/2\lambda^2 & (1 - \lambda)/2\lambda^2 \\ 0 & 0 & 1 \end{pmatrix}$$

admits a minimal realization $W(\lambda) = I_3 + C(\lambda I_3 - A)^{-1}B$, where $B = I_3$ and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that $\delta(W) = 3$ and by the proof of the main result in [22], we get $\rho(W) = 4$. On the other hand, W is irreducible; it does not admit a non-trivial minimal factorization, so $\beta(W) = 3$. This implies that $\rho(W) - \beta(W) = 1$, while $\delta(W) - 1 = 2$. □

Another optimization problem involves the minimization of the sum of the reduced infeasibilities of the jobs. Here the *reduced infeasibility* of job j is given by

$$\mathscr{I}'_j = \max\{F(O_j^1) - S(O_j^2) - 1,\ 0\}.$$

That is, in this case an infeasibility of one time unit is not counted. The objective of this variant of 2MFSP is to find a schedule which respects the deadline D and minimizes $\sum_{j=1}^{k} \mathscr{I}'_j$. If J is an instance of this variant of 2MFSP, then the minimal value of the sum of the reduced infeasibilities for J is denoted by $v(J)$.

For a rational $n \times n$ matrix function W, the positive integer $\alpha(W)$ maximizes the number of factors appearing in a minimal factorization of W. In other words,

$$\alpha(W) = \max\{p \mid W = W_1 \cdots W_p;\ \delta(W) = \sum_{i=1}^{p} \delta(W_i)\}.$$

Observe that $\alpha(W) = 1$ if and only if W does not admit a non-trivial minimal factorization, and that $\alpha(W) = \delta(W)$ if and only if W admits complete factorization. The following lemma is taken from [7].

Lemma 6.5. *Let W be a companion based rational matrix function and let J be the associated instance of 2MFSP with deadline $D = \delta(W)$. Then*

$$\alpha(W) + v(J) = \delta(W).$$

We also state and prove the following lemma from [21].

Lemma 6.6. *Let W be a rational $n \times n$ matrix function. Then*

$$\alpha(W) \leqslant 2\delta(W) - \rho(W).$$

Proof. Write $\alpha = \alpha(W)$ and let $W = W_1 \cdots W_\alpha$ be a minimal factorization of W, then

$$\rho(W) \leqslant \sum_{i=1}^{\alpha} \rho(W_i) \leqslant \sum_{i=1}^{\alpha} (2\delta(W_i) - 1) = 2\delta(W) - \alpha(W).$$

This proves the lemma. □

From Lemmas 6.5 and 6.6, we may conclude that, if W is a companion based rational matrix function and J is the associated instance of 2MFSP with deadline $D = \delta(W)$, then

$$v(J) = \delta(W) - \alpha(W) \geqslant \rho(W) - \delta(W). \tag{6}$$

It is worth mentioning that whatever connection there might be between the integers $v(J)$ and $\rho(W)$, it needs to be of a non-trivial nature. This is due to the fact that the problem of finding $\rho(W)$ is solvable in polynomial time, while the problem of determining $v(J)$ is NP-hard [19].

As a consequence, the inequality in (6) will not be an equality for each companion based matrix function W and its associated instance J of 2MFSP. The latter is illustrated by the following example that also appears in [7].

Example 6.7. Consider the following companion based matrix function

$$W(\lambda) = \begin{pmatrix} 1 & 1/(\lambda+1)^4(\lambda-1)^6 \\ 0 & \lambda^3/(\lambda+1)(\lambda-1)^2 \end{pmatrix}$$

with McMillan degree $\delta(W) = 10$. It is not difficult to see that the associated instance J of 2MFSP contains three jobs, namely the jobs $(s_1, t_1) = (3, 4)$, $(s_2, t_2) = (4, 6)$, and $(s_3, t_3) = (3, 0)$. Furthermore, the instance J has $D = \delta(W) = 10$.

By means of Johnson's Rule it has been established in [7], Example 2, that $\beta(W) = 3$. This implies $\rho(W) - \delta(W) = \beta(W) - 1 = 2$. On the other hand, Example 3 in the same paper shows that $v(J) = 3$. In this manner, we get $v(J) > \rho(W) - \delta(W)$.

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